

Boolos-style proofs of limitative theorems

György Serény*

Department of Algebra, Budapest University of Technology and Economics,
1111 Stoczek u. 2. H ép. 5. em., Budapest, Hungary

Key words: Limitative results of logic, Boolos's incompleteness proof, Berry's paradox

MSC (2000): 03F30

Boolos's proof of incompleteness is extended straightforwardly to yield simple "diagonalization-free" proofs of some classical limitative theorems of logic.

In his famous paper announcing the incompleteness theorem, Gödel remarked that, though his argument is analogous to the Richard and the Liar paradoxes, "Any epistemological antinomy could be used for a similar proof of the existence of undecidable propositions." ([7] Note 14). It is interesting that, despite the fact that the soundness of arguments like Gödel's one built on self-reference (or diagonalization) was often questioned (of course, from a philosophical not a mathematical point of view), the first attempt to support Gödel's claim and prove the theorem using another paradox (and hence without recourse to diagonalization) came only recently. In 1989, formalizing the Berry paradox consisting in the fact that *the least integer not nameable in fewer than nineteen syllables* has just now been named in eighteen syllables, G. Boolos proved the semantic version of the incompleteness theorem to the effect that there are arithmetical sentences that are true but unprovable in Peano arithmetic (see [3]). The proof, as Boolos notes at the end of his paper, "unlike the usual one, does not involve *diagonalization*". Not much later, in a letter, he adds "What strikes the author as of interest in the proof via Berry's paradox is [...] that it provides a *different sort of reason* for the incompleteness [...]" (cf. [4]).

Perhaps Boolos's proof was one of the factors that have inspired a wave of "proving old results in a new way" (see e.g. [1] and the references given there). Nevertheless, unlike the proof theoretical methods used in both Gödel's original proof and Boolos's one, most of these new proofs apply sophisticated model theoretical methods that can hardly be considered "finitistic". On the other hand, Boolos's proof can straightforwardly be extended to yield simple proofs of some fundamental theorems that are related very closely to the incompleteness theorem and to each other. The two versions of Gödel's first incompleteness theorem (the semantical and syntactical one describing respectively the relation between truth and provability and that of provability and refutability) together with their strengthening (the Gödel–Rosser theorem), Church's theorem on the undecidability of provability, and Tarski's theorem on the undefinability of truth, in a sense, constitute a complete circle of mutually related statements answering some basic questions on provability and truth. The close connection between these fundamental results is also witnessed by the fact that their standard proofs have essentially the same structure: they all can be derived from a general formal version of the Liar paradox, that is, they can be considered as different formal resolutions of this paradox (cf. [11]). Now, as we shall show below, almost the same can be said if we replace the Liar paradox by Berry's one. Actually, without any essential modification, the idea underlying Boolos's proof of incompleteness can be used to provide "diagonalization-free" proofs of all the basic limitative theorems mentioned above.

After fixing notation and giving the definition of basic notions, we first mimic Boolos's proof in a slightly more detailed form than that in which the original proof was given so that we can continue the proof in different directions, which is just what we shall do.

Let us first fix any one of the standard first order languages of arithmetic. By a formula (resp. sentence, term etc.) we mean a formula (resp. sentence, term etc.) of this language. Theories are arbitrary sets of sentences. Robinson arithmetic (cf. [8] I.1.1) will be denoted by Q . We shall denote the standard model of Q (as well as its universe) by ω , and say that a sentence is true (resp. a set is definable, defined etc.) if the sentence considered is true (resp. the set is

* email: sereny@math.bme.hu

definable, defined etc.) in ω . The variables are $v_0, v_1, \dots, v_i, \dots$. If it seems necessary to indicate the difference between the closed terms $0, s0, ss0, \dots$ and their values in ω (i.e. the natural numbers $0, 1, 2, \dots$), we shall denote the terms by the underlined versions of their values, but as a rule, since there is no danger of confusion, we omit the underlining. Generally, the value of a closed term in ω will be denoted by the boldface version of the letter denoting the term concerned. Let us choose one of the standard Gödel numberings. For any formula μ , $\ulcorner \mu \urcorner$ will denote the Gödel number of μ . “iff” stands for “if and only if”, and we often use the symbol “ \doteq ” to stress that the equality concerned is a definition.

We say that a formula is Σ_1 if it is of the form $(\exists v_i) \mu$ for some Δ_0 (i.e. bounded) formula μ . Σ_1 relations are those definable by a Σ_1 formula. A formula is Σ (or a Σ formula) if it is provably equivalent in Q to some element of the smallest set (i.e. the intersection of all sets) containing all Δ_0 formulas and being closed under conjunction, disjunction, existential quantification, and bounded universal quantification. Further, a formula is said to be Δ (or a Δ formula) if both the formula itself and its negation are Σ . A Σ (Δ) sentence is a Σ (Δ) formula that is a sentence. Clearly, a Σ_1 formula is also a Σ formula. A relation is called to be Σ (Δ) (or a Σ (Δ) relation) if it is definable by a Σ (Δ) formula. It can easily be checked that a relation is Σ iff it is Σ_1 (i.e. recursively enumerable), and a relation is Δ iff it is Δ_1 (i.e. recursive), cf. e.g. [12] p. 10. A straightforward induction on the complexity of formulas analogous to those that can be found in [5] (p. 25) and [8] (I.1.8) shows that Q is Σ complete, that is, all true Σ sentences are provable in Q .

Definition

- (i) For any term or formula e , let us denote by $|e|$ the number of symbols occurring in e (we shall call this number the *length* of e), and let $f : \omega^2 \rightarrow \omega$ be a recursive function such that, for any formula μ and natural number i , $f(i, \ulcorner \mu \urcorner) = \ulcorner (\forall v_0)(\mu \iff v_0 = i) \urcorner$. (Obviously, there exists such a function.)
- (ii) For any theory S , let us denote by $\mathcal{P}r_S$ the set of Gödel numbers of sentences provable in S .
- (iii) Let T be an arbitrary theory. Let us define the relations $\mathcal{F}m \subseteq \omega$, and $\mathcal{L}h, \mathcal{N}m \subseteq \omega^2$ as follows:

$$\mathcal{F}m \doteq \{i \in \omega : i = \ulcorner \mu \urcorner \text{ for some formula } \mu \text{ with at most one free variable } v_0\},$$

$$\mathcal{L}h \doteq \{(i, j) \in \omega^2 : i = \ulcorner \mu \urcorner \text{ for some formula } \mu \text{ such that } |\mu| < j\},$$

$$\mathcal{N}m \doteq \{(i, j) \in \omega^2 : j \in \mathcal{F}m \text{ and } f(i, j) \in \mathcal{P}r_T\}.$$

For any formula μ and number i , if $(i, \ulcorner \mu \urcorner) \in \mathcal{N}m$, that is, if $\mu = \mu(v_0)$ has at most one free variable v_0 and $T \vdash (\forall v_0)(\mu(v_0) \iff v_0 = i)$, then we say that the formula μ *names* the number i .

- (iv) It follows from the definition of Gödel numbering that the Gödel numbers of formulas whose variables are all among the first ones are bounded by a recursive function of their length. More precisely, in the case of any Gödel numbering, there is a recursive function g (depending on the particular Gödel numbering that has been chosen) such that, for any formula μ and number j , whenever all the variables of μ are among the first j ones (that is, for $j \geq 1$, they are all in the set $\{v_0, v_1, \dots, v_{j-1}\}$), $|\mu| < j$ implies that $\ulcorner \mu \urcorner < g(j)$.¹ Now, let us choose such a g , and let the relation $\mathcal{B} \subseteq \omega^2$ be defined in the following way:

$$\mathcal{B} \doteq \{(i, j) \in \omega^2 : (\ulcorner \mu \urcorner, j) \in \mathcal{L}h \text{ and } (i, \ulcorner \mu \urcorner) \in \mathcal{N}m \text{ for some formula } \mu \text{ such that } \ulcorner \mu \urcorner < g(j)\}.$$

¹For example, let us consider the most commonly used Gödel numbering, which (assuming that the Gödel numbers of primitive symbols of the language concerned have already been given) is defined for any sequence of symbols as follows: $\ulcorner \langle s_0, s_1, \dots, s_j \rangle \urcorner = p_0^{\ulcorner s_0 \urcorner} \cdot p_1^{\ulcorner s_1 \urcorner} \cdot \dots \cdot p_j^{\ulcorner s_j \urcorner}$, where p_i is the i th prime (see e.g. [10] pp.135–6). Apart from variables, our language has only finitely many primitive symbols, so we can define c to be any number that is greater than the Gödel numbers of primitive symbols except variables. Let $h(j) \doteq \max \{c\} \cup \{\ulcorner v_i \urcorner : i \leq j\}$ for every j . Then, for any formula μ such that $|\mu| < j$ and all the variables of μ are among the first j ones, $\ulcorner \mu \urcorner < p_j^{h(j) \cdot j}$. Clearly, the function $g(j) = p_j^{h(j) \cdot j}$ is recursive.

(v) Obviously, $\mathcal{F}m$ and $\mathcal{L}h$ are Δ_1 relations. Let us suppose that $\mathcal{P}r_T$ is definable. (This condition is obviously satisfied if, e.g., the set of Gödel numbers of sentences belonging to T is itself definable.) Then $\mathcal{N}m$ and \mathcal{B} are also definable.

(a) Let $\varphi(v_0, v_1)$ be a formula (with at most the free variables v_0, v_1) defining the relation \mathcal{B} . We shall choose φ to be Σ_1 whenever T is recursively axiomatizable. This is possible since, in this case, $\mathcal{P}r_T$ is Σ_1 , thus both $\mathcal{N}m$ and \mathcal{B} are also Σ_1 . (Recall that the class of recursively enumerable relations is closed under intersection, existential quantification, and the substitution of recursive functions, cf. e.g. [12] pp. 27–8).

Note that, if a formula μ has at most one free variable v_0 and $|\mu| < j$, then, by renaming the bound variables of μ , we can obtain a formula μ^* such that μ^* has at most one free variable v_0 , μ and μ^* are provably equivalent in Q , $|\mu^*| = |\mu|$, and all the variables of μ^* are among the first j ones, that is, according to our remarks above, $\ulcorner \mu^* \urcorner < g(j)$. In view of this fact, for any number i and closed term s , $\varphi(i, s)$ is true iff there is a formula μ such that $|\mu| < s$ and μ names the number i .

(b) Let $\psi(v_0, v_1) \doteq \neg\varphi(v_0, v_1) \wedge (\forall v_2 < v_0)\varphi(v_2, v_1)$.

For any number i and closed term s , $\psi(i, s)$ is true iff i is the least natural number that cannot be named by a formula of length $< s$. (Clearly, $\psi(v_0, s)$ has at most one free variable v_0 .)

(vi) Let $k_1 \doteq |\psi(v_0, v_1)|$ and let k_2 be any natural number that is greater than the number of free occurrences of v_1 in $\psi(v_0, v_1)$. Let $k \doteq k_1 \cdot k_2$, $t \doteq \underline{10} \cdot (\underline{k} \cdot \underline{k})$. Then $k \geq k_1 > 3, k \geq k_2 \geq 1$.

(vii) If T is a consistent extension of Q , then every formula can name at most one number. (Indeed, $i \neq j$ implies $Q \vdash i \neq j$, cf. [8] I.1.6(3).) Further, clearly, formulas provably equivalent in T name the same number (if they name a number at all). Finally, up to provable equivalence in T , there are only finitely many formulas of less than a given length having at most one free variable v_0 . (Recall that, apart from variables, our language has only finitely many primitive symbols and see our remarks in (v)(a).) Consequently, there are only finitely many different numbers that can be named by formulas of less than a given length. Thus, there is a least number that cannot be named by a formula of length less than \mathbf{t} . Let it be denoted by n .

Theorem

If T is a consistent extension of Q and $\mathcal{P}r_T$ is definable, then $\psi(n, t)$ is true, but $T \not\vdash \psi(n, t)$.

Proof. By definition, n is the least number that cannot be named by a formula of length $< \mathbf{t}$, and, again by definition, $\psi(n, t)$ is true just in this case. Consequently,

(1) $\psi(n, t)$ is true.

On the other hand, by the definition of ψ , (1) implies that

(2) $\varphi(n, t)$ is false.

Further, it is easy to see that

(3) if $T \vdash \psi(n, t)$, then $\psi(v_0, t)$ names the number n .

Actually, we have to show that $T \vdash \psi(n, t)$ implies $T \vdash (\forall v_0)(\psi(v_0, t) \iff v_0 = n)$. Clearly, in one direction, the formal implication is trivial: $T \vdash \psi(n, t) \wedge v_0 = n \implies \psi(v_0, t)$. The other direction, in turn, follows from the fact that T is an extension of Q since $Q \vdash v_0 \leq i \vee i \leq v_0$ (cf. [8] I.1.6 (5)), which, in turn, implies a weak kind of provable uniqueness of least elements; more precisely, for any formula $\mu(v_0)$ and number i ,

$$Q \vdash \neg\mu(i) \wedge (\forall v_2 < i)\mu(v_2) \implies (\forall v_0)[\neg\mu(v_0) \wedge (\forall v_2 < v_0)\mu(v_2) \implies v_0 = i].$$

Now, it follows from $k > 3$ that $18k < 8k^2$. Thus the definition of t implies that $|\psi(v_0, t)| \leq |\psi(v_0, v_1)| + k_2|t| = k_1 + k_2(15 + k + 1 + k + 1) = k_1 + k_2(17 + 2k) \leq k + k(17 + 2k) = 18k + 2k^2 < 10k^2 = \mathbf{t}$. So we have $|\psi(v_0, t)| < \mathbf{t}$, which, together with (3), shows that, if $T \vdash \psi(n, t)$, then $\psi(v_0, t)$ is actually a formula witnessing the truth of $\varphi(n, t)$. But $\varphi(n, t)$ is false by (2). Consequently, $T \not\vdash \psi(n, t)$.

Now we can give the semantical incompleteness theorem in the usual formulation. The theory T is called to be *sound* if all the sentences belonging to T are true.

Corollary 1 (Semantic version of Gödel's first incompleteness theorem)

Let \mathcal{Pr}_T be definable (in particular, let T be recursively axiomatizable). If T is sound, then T is incomplete.

Proof. First of all, for any theory S , let us denote by $\text{Ded } S$ the set of all sentences provable in S , and let $T' \doteq Q \cup T$. Then obviously, $\text{Ded } T' = \text{Ded } (Q \cup \text{Ded } T)$, that is, $\mathcal{Pr}_{T'} = \mathcal{Pr}_{Q \cup \text{Ded } T}$. Since Q is finite and \mathcal{Pr}_T is definable by our assumption, the set of Gödel numbers of the sentences in $Q \cup \text{Ded } T$ is again definable, which, in turn, implies the definability of the set $\mathcal{Pr}_{Q \cup \text{Ded } T} = \mathcal{Pr}_{T'}$. Further, Q is sound by definition, thus T' is also sound. Soundness, in turn, implies consistency. Consequently, T' satisfies the conditions of the Theorem. Therefore $T' \not\vdash \psi(n, t)$ and $\neg\psi(n, t)$ is false. Thus, on the one hand, $T \not\vdash \psi(n, t)$ follows from the fact that $T \subseteq T'$, on the other hand, sound theories cannot prove false sentences.

So far we have only reiterated Boolos's proof with some minor modifications that open up the possibility to make a few steps farther along the lines set by the original proof, and have formulated its most immediate consequence.² In order to proceed, let us observe that, though Boolos's proof is essentially a formalization of the Berry paradox, it is not the most straightforward one. As a matter of fact, the theorem that can be considered as the most faithful formal version of the Berry paradox is Tarski's theorem on the undefinability of truth. Rephrasing Gödel's above quoted remark, we may conjecture that "The formal version of any epistemological antinomy is just the statement on the undefinability of truth, and hence could be used for its proof". The reason is simple enough. As Tarski puts it in connection with the Liar paradox (cf. [13] p. 76.), we cannot talk about the truth in the language of arithmetic since otherwise "the antinomy of the liar could actually be reconstructed in this language". As a simple corollary of the Theorem shows, literally the same can be said about the Berry paradox.

Corollary 2 (Tarski's theorem on the undefinability of arithmetical truth)

The set of Gödel numbers of true sentences is not definable.

Proof. Let \mathcal{Tr} be the set of Gödel numbers of true sentences and let us suppose that \mathcal{Tr} is definable. Choose T in the Theorem to be the set of all true sentences, that is, let $T = \{\sigma : \sigma \text{ is a true sentence}\}$. Then, clearly, T is deductively closed, i.e. the sentences provable in T are all in T . Consequently, $\mathcal{Pr}_T = \mathcal{Tr}$, so that T is consistent and \mathcal{Pr}_T is definable. Moreover, by definition, $T \supseteq Q$. So we can apply the Theorem: $\psi(n, t)$ is true but unprovable in T . But $\mathcal{Pr}_T = \mathcal{Tr}$ implies that this is impossible because it means that, for any sentence σ , σ is true iff σ is provable in T .

²The detailed exposition, however, has its reward. The Theorem is a slightly more general version of the *semantical* incompleteness theorem than the usual one. Indeed, it seems that the standard proofs (cf. e.g. [6] p.229 or [2] p.100), being essentially based on the diagonal lemma in one way or other, yield the theorem in such a form in which the condition of soundness of the theory concerned (which is, of course, a much stronger requirement than that of its consistency) inevitably appears; see the proof of the abstract version of this theorem in [11]. For that matter, if we had followed Boolos's proof word by word, then we could have weakened even the condition that T is an extension of Q . Actually, in order to define n , it is enough to suppose that the sentences $\underline{i} \neq \underline{j}$, $i, j \in \omega$ are all theorems of T . Then we can proceed as follows. $\psi(v_0, t)$ defines n as the least element of a non-empty set of natural numbers. Since the least element of such a set is unique, the sentence $\eta \doteq (\forall v_0)(\psi(v_0, t) \iff v_0 = n)$, expressing the uniqueness of this element and the fact that this element is just n , is true. On the other hand, the provability of η in T would entail that $\varphi(n, t)$ is true (recall that $|\psi(v_0, t)| < \mathfrak{t}$), contradicting the truth of $\psi(n, t)$. As far as the third condition of the Theorem is concerned, the usual strong assumption of recursive axiomatizability of T can obviously be weakened to the definability of \mathcal{Pr}_T in the classical proofs as well.

Kikuchi has modified Boolos's notion of naming to obtain the syntactic version of the first incompleteness theorem for suitable extensions of Peano arithmetic and the second incompleteness theorem (see [9]).³ As far as the first incompleteness theorem is concerned, in fact, this modification is not needed. What is more important, with the help of making some plausible additional observations, we can derive the syntactic incompleteness theorem from the previous results for a considerably weaker theory than Peano arithmetic:

Corollary 3 (Syntactic version of Gödel's first incompleteness theorem)

Let T be a recursively axiomatizable extension of Q .

- (i) *If T is consistent, then $T \not\vdash \neg\varphi(n, t)$.*
- (ii) *If T is ω -consistent, then $T \not\vdash \varphi(n, t)$.*

Proof. Since T is recursively axiomatizable, $\varphi(v_0, v_1)$ is, by definition, a Σ_1 formula. Further, since ω -consistency implies consistency, the conditions of the Theorem hold in both cases.

(i) Since $\varphi(v_2, t)$ is Σ , the sentence $(\forall v_2 < n)\varphi(v_2, t)$ is also Σ . Moreover, it is true. (This follows from the definition of ψ and (1) in the proof of the Theorem.) Therefore, by Σ completeness, $T \vdash (\forall v_2 < n)\varphi(v_2, t)$. Hence $T \vdash \neg\varphi(n, t)$ would imply $T \vdash \psi(n, t)$, contradicting the Theorem. Thus $T \not\vdash \neg\varphi(n, t)$.⁴

(ii) Suppose that T is ω -consistent. Since $\varphi(v_0, v_1)$ is now Σ_1 , there is a Δ_0 formula $\mu(v_0, v_1, v_i)$ such that $\varphi(v_0, v_1) = (\exists v_i)\mu(v_0, v_1, v_i)$. As we have already seen, $\varphi(n, t)$ is false (cf. (2) in the proof of the Theorem). It follows from this that, for any number j , $\neg\mu(n, t, j)$ is a true Δ_0 sentence. Using Σ completeness, we have $T \vdash \neg\mu(n, t, j)$ for every j , which, by the definition of ω -consistency,⁵ implies that $T \not\vdash (\exists v_i)\mu(n, t, v_i)$, i.e. $T \not\vdash \varphi(n, t)$.

One of the standard ways to prove the Gödel–Rosser theorem is to show that it is a direct consequence of the Church theorem. We shall also follow this route, that is, using the previous results, we first show that no consistent extension of Q is decidable:

Corollary 4 (Church's theorem on the undecidability of arithmetic)

If T is a consistent extension of Q , then T is undecidable.

Proof. Suppose, for sake of contradiction, that $\mathcal{P}r_T$ is a recursive relation. It follows from this that $\mathcal{N}m$ is also a recursive one, hence \mathcal{B} is again recursive since, on the one hand, $\mathcal{F}m$ and $\mathcal{L}h$ are recursive, on the other, the class of recursive relations is closed under intersection, bounded quantification, and the substitution of recursive functions (cf. e.g. [12] pp. 27–8). Consequently, the formula φ defining \mathcal{B} can now be chosen to be Δ . Since T is supposed to be a consistent extension of Q and $\mathcal{P}r_T$ to be recursive (which, of course, implies the recursive axiomatizability of T), we can apply Corollary 3 (i). Consequently, $T \not\vdash \neg\varphi(n, t)$. This, however, leads to a contradiction since, by (2) in the proof of the Theorem, $\neg\varphi(n, t)$ is true, i.e. it is a true Σ sentence. Its truth, in turn, by Σ completeness, implies its provability in T , that is, $T \vdash \neg\varphi(n, t)$.

³There is, however, a minor mistake in his proof of the first incompleteness theorem (see the proof of Theorem 2.2 (ii) in [9]). Indeed (using the notation of [9]), $Q(m, \rho)$ is obviously *not* Σ_1 . What is needed, therefore, in order for *that* proof given in [9] to go through, is the simple fact (to be shown, needless to say, without using the soundness of PA) that there is a Σ_1 sentence $Q^*(m, \rho)$ satisfying not only the requirement that $Q(m, \rho)$ implies $Q^*(m, \rho)$ in PA , *but also* the additional one that the truth of $Q^*(m, \rho)$ implies the same for $Q(m, \rho)$.

⁴Note that $\neg\varphi(n, t)$ is yet another sentence that is true but unprovable.

⁵ T is ω -consistent if for any formula $\eta(v_i)$, it follows from $T \vdash (\exists v_i)\eta(v_i)$ that $T \not\vdash \neg\eta(j)$ for some number j (cf. e.g. [10] p. 142).

In the usual way, Church's Theorem immediately yields

Corollary 5 (Rosser–Gödel incompleteness theorem)

If T is a consistent and recursively axiomatizable extension of Q , then T is incomplete.

Proof. Let us suppose that, on the contrary, T is complete. Let $\text{Snt}(v_0)$ and $\text{Neg}(v_0, v_1)$ denote Δ formulas defining, respectively, the set of Gödel numbers of sentences and the relation that holds between the Gödel number of a sentence and that of its negation. Further, let $\text{Pr}_T(v_0)$ a Σ formula defining the set \mathcal{Pr}_T . Now we set

$$\text{Pr}_T(v_0) \doteq \neg \text{Snt}(v_0) \vee (\exists v_1)(\text{Pr}_T(v_1) \wedge \text{Neg}(v_0, v_1)).$$

Then $\text{Pr}_T(v_0)$ is a Σ formula. On the other hand, it follows from the completeness and consistency of T that $\text{Pr}_T(v_0)$ defines just the complement of \mathcal{Pr}_T . Therefore, both \mathcal{Pr}_T and its complement are Σ , so that \mathcal{Pr}_T is recursive, contradicting the previous corollary.

The proofs we have given demonstrate that the Boolos-style formalization of Berry's paradox is, in fact, a proof *schema*. Indeed, in order to obtain the proofs of *Gödel's semantical incompleteness theorem*, *Tarski's theorem*, *Gödel's syntactical incompleteness theorem*, and *Church's theorem*, we have simply applied the *common* conceptual framework given implicitly by Boolos's incompleteness proof to four kinds of formal theories of arithmetic, namely, to theories for which the set \mathcal{Pr}_T is, respectively, *definable*, *the set of (Gödel numbers of) all true sentences*, *recursively enumerable*, and *recursive*.

Acknowledgment

The research was supported by Hungarian NSF grants No. T43242, T30314, and T035192.

References

- [1] Z. Adamowicz, and T. Bigorajska, Existentially closed structures and Gödel's second incompleteness theorem. *J. Symbolic Logic* **66**, 349–356(2001).
- [2] D.W. Barnes and J.M. Mack, *An Algebraic Introduction to Mathematical Logic* (Springer–Verlag, New York, 1975).
- [3] G. Boolos, A new proof of the Gödel incompleteness theorem. *Notices Amer. Math. Soc.* **36**, 388–390(1989).
- [4] G. Boolos, A letter from George Boolos. *Notices Amer. Math. Soc.* **36**, 676(1989).
- [5] G. Boolos, *The Logic of Provability* (Cambridge University Press, Cambridge, 1995).
- [6] H. B. Enderton, *A Mathematical Introduction to Logic* (Academic Press, New York, 1972).
- [7] K. Gödel, On Formally Undecidable Propositions of *Principia Mathematica* and Related systems I. In: *Gödel's Theorem in Focus* (S. G. Shanker, ed., Routledge, London 1988).
- [8] P. Hájek, and P. Pudlák, *Metamathematics of First–Order Arithmetic* (Springer, Berlin 1993).
- [9] M. Kikuchi, A note on Boolos' proof of the incompleteness theorem. *Math. Logic Quarterly* **40**, 528–532(1994).
- [10] E. Mendelson, *Introduction to Mathematical Logic* (D. Van Nostrand Company, Princeton 1964).
- [11] G. Serény, Gödel, Tarski, Church, and the Liar. *The Bulletin of Symbolic Logic* **9**, 3–25(2003).
- [12] R. M. Smullyan, *Recursion Theory for Metamathematics* (Oxford Univ. Press, New York, 1993).
- [13] A. Tarski, Truth and proof. *Scientific American* **220**, No 6, 63–77(1969).